A result from Furutsu-Novikov theorem

Consider an N-dimensional system with the Hamiltonian . We can write down the equations of motion by Langevin dynamics

 ,

where  is a diagonal friction matrix with positive elements, , and  is the white noise normal random force.

Note that the density distribution , where  denotes the average over the stochastic process. What we will do here is just to deal with this term . First, we will make use of Furutsu-Novikov theorem[1],



where  is a functional of ,  is the i-th component of . The correlation of the components is

 .

For the variational derivative, we will use the equality[2]

 .

Here  means the k-th diagonal element of , and  means the initial time. Then we obtain

 .

Substitute Eq. and Eq. into Eq. , we have



So, we obtain

 ,

which is the result we want.

Reference

[1] E. A. Novikov, *J. Exptl. Theoret. Phys.* **20** (5), 1290-1294, 1965.

[2] V. I. Klyatskin, *Stochastic Equations: Theory and Applications in Acoustics, Hydrodynamics, Magnetohydrodynamics, and Radiophysics, Volume 1*, Chapter 10: *Gaussian Random Field Delta-Correlated in Time (ordinary differential equations)*.